POWER OPTIMIZATION FOR THE PROPULSION OF LIGHTWEIGHT VEHICLES

Analysis and Application to Pedal-Electric Bicycle

SUMMARY

The equations for the power required to move an object in air are such that the mechanical energy consumed when a vehicle covers a given course in a given time depends on how the propulsion power is regulated to accommodate changing load conditions. In this article we explore strategies for minimizing that energy. Mathematical simulations of road trips that include hills, winds and stops reveal that contrary to common assumptions, the total energy is minimized when maximum available power is high enough to enable rapid accelerations and elevation gains.

Our principal application is the human-electric hybrid bicycle, *a.k.a.* e-bike or *pedelec*, which is propelled by two distinct energy sources with fundamentally different characteristics: a power-limited source (the rider) and an energy-limited source (the battery). The instantaneous power drawn from each source is dynamically adjusted in response to changing load conditions. Our objective is to apportion muscle power and electric power in such a way that a given course is completed in the shortest time possible with the least amount of electric energy. We make the distinction between reactive and resistive mechanical loads, and show that electric power should be applied in proportion to the instantaneous ratio between the two.

The unique aspect of our analysis is the inclusion of the *reactance* of the mechanical load as a control variable in the power management algorithm. While this strategy is generally aimed at minimizing the output energy, the algorithm allows for a predefined relationship between motor efficiency and motor power to accurately account for energy losses in the motor as well. Thermal losses in the battery and controller are also included. Finally, special consideration is given to power management strategies with and without regenerative braking.

DESCRIPTION OF THE MODEL

The energy required for a given scenario is calculated with the use of a simulation model. Input parameters are physical constants such as air density, gravity, etc; rider and vehicle data such as weights, areas, coefficients of tire rolling resistance and air drag, etc; and route information. The route information includes the elevation profile, locations of speed restrictions or stops, wind speed and wind direction along the course. The model is applicable to motorized and human powered vehicles. In the human-powered case including e-bikes, the rider's instantaneous power is estimated from the power history looking back and the power demand looking ahead. Output parameters include instantaneous and average speed and remaining battery capacity as a function of time and distance. The smallest battery capacity that is sufficient for achieving a given average speed for the entire course, and the corresponding muscle and motor power levels as a function of distance, are computed by means of successive iterations. Account is taken of the effect of vehicle speed on the apparent wind velocity, and consequently on the effective drag-area and drag coefficient (which, in turn, affects vehicle speed due to limited power).

The general goal is to cover a long distance in a short time using little electric energy. The premise of the optimization, whether it is for maximum range, minimum time or minimum energy, is to ration the propulsion power depending on how efficiently it will be used against the mechanical load prevailing at any time. For this purpose, *efficiency* and *mechanical load* are defined in a specific context. These and other technical terms are given below for clarity.

DEFINITIONS

Energy and Power

Energy is work, and *power* is the time rate at which work is done. The term 'energy' is also applied to a system's ability to do work. For example, a vehicle at a high elevation contains potential energy, and a vehicle moving at high speed contains kinetic energy. Heat is a form of energy. Electric power is voltage times current. Mechanical power is force times speed, and mechanical energy is force times distance. Common units of power and energy are watts and watt-hours, respectively.

Energy is a precious commodity, but power is generally a choice of convenience. *Power has no intrinsic cost, and does not increase the unit cost of energy.*

Load

Reaction forces that oppose the propulsion force are called mechanical loads, or *loads* for short. Loads impede the mobility of the vehicle. Some loads (e.g., inertia) have the capacity to store energy temporarily; others (e.g., air drag) do not.

Efficiency

In the most general context, efficiency is defined as the ratio between output power and input power. But the demarcation between input and output depends, to a large extent, on our intent. For example, a light bulb that is not efficient can be efficient as a heat source. Thus, in a more specific context, efficiency is the ratio of *useful power* to input power.

For a personal transportation vehicle, some of the output power can be characterized as useful power or alternately, as waste. In this paper we describe a power management system that seeks efficiency in a specific context. For this purpose, we define reactive loads and resistive loads in the mechanical domain.

Reactive and resistive loads

A load is said to be reactive if the energy that enters it is temporarily stored; and resistive if it is immediately dissipated. For example, *hill force* is a reactive load because the energy used for gaining elevation is stored as potential energy. In contrast, *air drag* is a resistive load because the energy used for displacing air is immediately dissipated. Typically, reactive and resistive loads are present at the same time. Sometimes one load type dominates over the other.

The total mechanical load is always equal to the propulsive force (traction at the driven wheel); therefore its magnitude is controlled by the power applied at the source. But on a road course with hill climbs, descents, accelerations, headwinds, cross winds and tailwinds, even if the magnitude of the load were kept constant, load reactance and load resistance would be changing. The unique aspect of our power management algorithm is the strategy of increasing the power when the mechanical load becomes reactive, and decreasing it when the mechanical load becomes resistive. Except when winds are unusually strong, this strategy results in traveling at nearly constant speed regardless of load conditions (cruise control). This strategy, which may be described in engineering terms as use of a low-impedance source, requires that the maximum power must be relatively high.

The instantaneous mechanical load, “as seen” by the source, can be modeled as a circuit comprised of the following physical components:

- Hill force (product of weight and incline)
- Inertia force (product of mass and acceleration)
- Air drag (proportional to the square of the forward airspeed) –Eq. 25
- Tire rolling resistance (proportional to groundspeed) –Eq. 13

The first two of these loads are reactive; the last two are resistive. In this article, *incline* is defined as the rate of elevation change per distance, which is slightly different than the standard definition of highway grades. Except for unusually steep grades the difference is very small.

The terminology of reactive and resistive load impedances is borrowed from AC electrical circuits and is used here for explanation by analogy. In an electrical circuit made of only resistive loads, such as resistors and incandescent light bulbs, all the energy that enters the load circuit is immediately dissipated. But in reactive loads -- capacitors and inductors --, a portion of the energy is temporarily stored. This energy is eventually transferred to resistive elements or returned to the source.

Similarly, when a vehicle is cruising at constant speed on a flat road where the only loads are air drag and tire rolling resistance, all the energy that enters the load circuit is immediately dissipated. But when hills and accelerations are involved, a portion of the propulsion energy is temporarily stored as potential and kinetic energy. This energy is eventually dissipated in resistive loads or returned to the source.

The “returning to the source” of the stored energy is not to be confused with regenerative braking. An example of how stored energy returns to the source, is the relative ease of pedaling a bicycle (or the ability to coast without slowing) when going downhill.

ANALYSIS AND OBSERVATIONS

By definition, any part of the course that is covered below average speed must be compensated by covering another part above average speed. Because air drag and tire rolling resistance increase with speed, this “make-up speed” always requires more energy per distance, than the energy that was saved by going slowly. Therefore, falling below average speed always costs time and energy. These phenomena are quantified with numerical examples below.

TIME AND ENERGY COSTS DUE TO SPEED FLUCTUATIONS

Time cost

We start with a basic question: if the first half of a distance is covered at 10 mph and the second half at 30 mph, what is the average speed over the total distance?

For convenience we may consider, for example, a 2-mile long distance. Accordingly, covering the first mile at 10 mph would take 1/10 hour (6 minutes), and covering the second mile at 30 mph would take 1/30 hour (2 minutes). Thus, it takes 8 minutes to cover 2 miles, averaging 4 minutes per mile, or 15 miles per hour.

This answer may be surprising at first, but it demonstrates that below-average speeds have greater influence on the average speed than above-average speeds do.

More generally, if 1/n of a distance is covered at speed u_1 and the remaining part at speed u_2, the resulting average speed u_{avg} can be calculated using the following formula:

$$u_{avg} = (u_1 * u_2) / [1/n * u_2 + (1 – 1/n) * u_1]$$

For the numerical example above, we substitute n = 2; u_1 = 10 mph; u_2 = 30 mph. Thus,

$$u_{avg} = (10 * 30) / [0.5*30 + (1-0.5)*10] = (300) / [15 + 5] = 300 / 20 = 15 \text{ mph.}$$

Clearly, if u_1 and u_2 result from symmetric deviations from an average speed, the highest average speed (shortest time) is achieved when the deviations are as small as possible. The time cost will increase as the deviations increase.

Energy cost

The energy penalty for riding below average speed is even more severe than the time penalty, because resistive loads increase with speed. Particularly, if there is no wind, air drag increases in proportion to squared speed (Equation 1). Consequently, although traveling at a low average speed *saves* energy (except if that means dragging the brakes), traveling below average speed *wastes* energy.

To quantify this waste of energy, we can use the same numerical example. We are comparing two scenarios (Refer to Tables 1 and 2). Scenario 1 (base line, or reference scenario) is to cover the first half of a distance at a constant 10 mph and the second half at a constant 30 mph. Scenario 2 is to ride both halves of the distance at constant 15 mph. To calculate the energies without specifying the values of physical constants such as frontal area, drag coefficient, tire rolling resistance, etc, we constrain the problem as follows: Since the total distance is arbitrary, we can define it such that at 10 mph the tire rolling energy at the half-way point becomes 100 Wh. We also know that tire rolling resistance increases as speed, and in the absence of wind, air drag increases as speed-squared; therefore at a particular speed, air drag power and tire rolling power will become equal. A specific combination of wind velocity, weight, tire and aerodynamic properties must exist, such that this particular speed is 10 mph. This value is chosen as an example because it is reasonable for a road bike. With this assumption we calculate the dissipated energies of the first and second scenarios, as shown in Tables 1 and 2, respectively.

SCENARIO 1 (Baseline)	1st half	2nd half	combined
average speed (mph)	10	30	15
Tire rolling energy (Wh)	100	300	400
Air drag energy (Wh)	100	900	1000
Total dissipated energy (Wh)	200	1200	**1400**
Total dissipated energy relative to baseline (Wh)	-	-	-

Table 1

SCENARIO 2 (economy mode)	1st half	2nd half	combined
average speed (mph)	15	15	15
Tire rolling energy (Wh)	150	150	300
Air drag energy (Wh)	225	225	450
Total dissipated energy (Wh)	375	375	750
Total dissipated energy relative to baseline (Wh)	175	-825	**-650**

Table 2

In Table 1, the 1st half and 2nd half tire rolling energies must have a 3:1 ratio because they are proportional to the corresponding speeds –see Equation 13. And the 1st half tire rolling energy is 100 Wh; thus, the 2nd half tire rolling energy is calculated as 300 Wh.

Similarly, air drag energies in Table 1 must have a 9:1 ratio because they are proportional to the corresponding *squared speeds* –see Equation 1. And the 1st half air drag energy is 100 Wh; thus, the 2nd half air drag energy is calculated as 900 Wh.

In Table 2, tire rolling energies are calculated by multiplying the corresponding values in Table 1 by the ratio of speeds:

1st half: 100 * (15 / 10) = 150 Wh

2nd half: 300 * (15 / 30) = 150 Wh

Similarly, air drag energies in Table 2 are calculated by multiplying the corresponding values in Table 1 by the *square* of the ratio of speeds:

1st half: 100 * $(15 / 10)^2$ = 225 Wh

2nd half: 900 * $(15 / 30)^2$ = 225 Wh

Thus, the total dissipated energies are 1400 Wh for Scenario 1 and 750 Wh for Scenario 2.

This example illustrates how much air drag energy is saved by riding at 15 mph all the time, instead of 10 mph for the first half and 30 mph for the second half. The inability to ride at constant speed may be caused, for example, by having to go uphill for the first half and downhill for the second half.

For Scenario 2, note that in the 1st half 175 Wh extra energy is used by going faster than Scenario 1, but in the second half 825 Wh less energy is used by going slower than Scenario 1. The net result is a total energy saving of 650 Wh (or 54% of the total energy) without lowering the average speed. This way of managing the power, which prioritizes energy saving over performance and range, may be called the "economy mode".

Conversely, if we use all of the 1400 Wh energy of Scenario 1, but apply it at a constant speed for the whole distance, we can increase the performance without compromising range. Indeed, that constant speed would be almost 22 mph! This solution was found by solving equations, but here we can simply verify that the solution is correct. To do that, we construct Table 3. Again, we only need to remember that tire rolling energy is proportional to speed; air drag energy is proportional to the *square of* speed, and that they are both 100 Wh at 10 mph. We enter the trial solution (21.93 mph, to be precise) as the 1st half and 2nd half speeds, fill out the rest of Table 3, and confirm that the total dissipated energy becomes 1400 Wh, which matches the value in Table 1. Thus, we have verified that 21.93 mph is indeed the correct solution.

SCENARIO 3 (performance mode)	1st half	2nd half	combined
average speed (mph)	21.93	21.93	**21.93**
Tire rolling energy (Wh)	219.3	219.3	438.5
Air drag energy (Wh)	480.7	480.7	961.5
Total dissipated energy (Wh)	700.0	700.0	1400.0
Total dissipated energy relative to baseline (Wh)	500.0	-500.0	0.0

Table 3

We have shown that when a constant speed is maintained in both halves of the distance, an average speed of 21.9 mph is achieved instead of the 15.0 mph baseline, without using more energy. This way of managing the power, which prioritizes the average speed, may be called the "performance mode".

By extrapolation, it is clear that we could combine the lower average speed (15 mph) of the economy mode with the higher total energy (1400 Wh) of the baseline, in which case, a greater distance would be covered. After all, Scenario 2 leaves us with 650 Wh of extra energy. This is more than half of the total energy used in base line case (Scenario 1). So, compared to Scenario 1, the range would be more than twice as long, and we would still have the same 15 mph average speed using the same 1400 Wh energy. This way of managing the power, which prioritizes the distance, may be called the "range-extending mode".

It may be noticed that Tables 1 – 3 do not include the calculation of the energy used against hill force (reactive energy). And hill force is the presumed root cause of unequal speeds in Scenario 1. But although the total energy calculations include only the resistive energy, they are accurate because, as long as the starting and ending elevations are equal, and as long as no braking is involved, the energy used for elevation gain (reactive energy) makes no contribution to the total energy. Reactive energy leaving the source during the climb would be exactly equal to the reactive energy returned to the source during the descent. This is not a simplifying assumption or approximation; it is exact. In fact, this "net zero" assessment for the climbing energy is more accurate than any calculations or measurements that we would be able to make. This subject is discussed later in this article under the heading "Energy Cost of Hills".

From these example scenarios we can see that slow climbs and fast descents (typical of unassisted pedaling due to the limitation of power) have a high cost in energy for given performance (Scenarios 1 and 2), and a high cost in performance for given energy (Scenarios 1 and 3).

We conclude that electric-assisted pedaling not only supplements the rider with more energy, but it can significantly increase the performance returned per unit of energy used, provided that the supplemental energy is rationed optimally.

This means that the electrical system and the rest of the vehicle should be designed and scaled to deliver the high power-to-weight-ratio necessary for climbing hills without slowing down. The same principle must also be applied during accelerations. Because battery capacity is limited, electrical assistance should be significantly reduced or turned off at high speeds, whereas human power should be sustained. By the same considerations, human power does not need to be as dramatically increased during hill climbs as might be optimal for unassisted pedaling.

In these examples we are calculating only the mechanical energy required at the wheel. In the simulation model, heat losses that occur at both the load side and the source side of the energy equation are also included. This subject is discussed later in the article under the "heat" heading.

We have seen that by the principle of conservation of energy, hill force and inertia force don't make a net contribution to the overall energy equation. But hills and accelerations still increase energy consumption indirectly, by dictating the speed of travel. These phenomena are discussed below.

HILLS

Elevation gain requires extra energy. But during the descent to the starting elevation, all of this extra energy is recovered. So, in the long term, by the principle of conservation of energy, the net energy consumed by elevation gain / loss will be zero; only the air drag and tire rolling energies need to be minimized. In other words, hill force is a reactive load, and hence, does not dissipate energy.

At first, the notion that ascending and then descending a hill is an energy-neutral activity seems to be at odds with observation. Of course, one gets more tired on a hillier course. To explain this phenomenon, we must discern *where* the energy loss occurs. For this purpose, we explore the path of the energy through reactive and resistive loads in a time trial that includes hills.

In a time trial without electric assistance, the athlete is going as fast as possible all the time. Therefore, when going uphill, speed is reduced; a relatively large portion of the athlete's energy is used against the reactive load (hill force) and stored as potential energy. During the descent back to the starting elevation, all this stored energy is returned to help the athlete; hence the descending speed is higher than the ascending speed. But because air drag and tire rolling resistance increase with speed, the amount of energy that was stored during the (slow) ascent is less than the extra energy dissipated during the (fast) descent. This energy deficit is covered by the athlete working out for a longer time due to the slower average speed. So, while it is true that a hilly time trial consumes more energy than a flat one of the same length, the extra energy is consumed during the descent. And provided that the starting and ending elevations are the same and no braking is involved, the total energy can be accurately computed from the speed profile with no regard to the elevation profile. In other words, if the uphill and downhill speeds and distances were reproduced on a flat course, the total energy would be exactly equal to that on the hilly course.

This has an important implication: It means that if the athlete had enough *power* to climb the hill at the same speed as descending it, then, a hilly time trial would have consumed exactly the same energy as a flat one of the same length and finishing time. An electric motor can enable the athlete to do just that, provided that it has enough *power*.

ACCELERATIONS

When accelerating, the inertia force requires power but does not dissipate energy. Hence, even if the ride includes a lot of accelerations and decelerations, inertia forces do not directly affect the total energy usage. But just like hills, accelerations lead to increased energy consumption indirectly. For example during constant acceleration from a standstill, air drag dissipates 2 times, and tire rolling resistance dissipates 4/3 times the energy needed to cover the same distance at the same time at constant speed (Equations 12 and 17, respectively).

Accelerating to a given speed

When bicycling in urban areas, average speed is reduced by slow accelerations due to the limitation of maximum available power. With an e-bike, accelerations can be faster because more power is available. But how much power should be applied? On one hand, accelerating over a given distance with a given average speed dissipates twice as much air-drag energy, and 1/3 more tire rolling energy, than constant-speed cruising the same distance at the same average speed, as stated above. On the other hand, getting up to speed more quickly will shorten the distance of the acceleration. So, for each acceleration (whether from a standstill or an initial speed), the same question must be asked: what is the optimal rate of acceleration to reach a given distance at a given time with the least amount of energy?

At first, it may appear to be self-evident that since the goal is to conserve energy, accelerations should be kept to a minimum. But considering the average speed requirement, we must be explicit about which aspect of acceleration is to be minimized: the rate of acceleration, the duration of the acceleration, the speed to accelerate to, or the distance to accelerate over. Clearly, it would not be possible to minimize all of them.

To emphasize the importance of the question, we turn to a common performance benchmark for automobiles, the time it takes to accelerate from 0 to 60 miles per hour. It is generally assumed that the quicker acceleration would require more energy. But in fact, the energy required to accelerate a car from 0 – 60 mph in 7 seconds (for example), is significantly *less* than the energy required to accelerate the same car from 0 – 60 mph in 14 seconds. The higher the rate of acceleration is, the lower is the required mechanical energy. Fuel consumption is a different matter if combustion efficiency declines with power, but that is not a universally valid assumption either.

The argument that a higher rate of acceleration would *save* energy, and might even save fuel in some cases, may seem counterintuitive at first. To explain, we note that because the beginning and ending speeds are the same, the work done against inertia is the same regardless of the rate of acceleration. Alas, the faster acceleration requires more propulsive force against inertia (Newton's second law), but that force is applied over a shorter time and distance, and thus does the same work. The only energy difference between accelerating quickly or slowly is due to the work done against the resistive loads (air drag and tire rolling resistance). With the quicker acceleration, these energy losses are smaller because the same drag forces are dragged a shorter distance. So, the total *mechanical* energy reaches a minimum when the rate of acceleration becomes maximum.

<u>Accelerating to a given distance</u>

At this point, one might argue that the more relevant contest would have been to accelerate to a given *distance* instead of a given speed. In fact, in our e-bike example we are interested in knowing which distance and speed we should accelerate to at which rate, before ending the acceleration and continuing at constant speed, to reach a given average speed at the end of a given distance.

Surprisingly enough, equations for that scenario show us that the minimum-energy solution is still the fastest acceleration to the shortest distance –and consequently cruising at the slowest speed, that is possible. This is explained below and illustrated in Figures 1, 2 and 3.

<u>Accelerating and then cruising</u>

One often accelerates from a stop to a constant cruising speed, and that speed is then maintained until the next stop. We want to minimize the energy of this whole process. We assume that the time t_{total} and distance d_{total} are given; our goal is to calculate the rate of the acceleration, and where and when to transition from accelerating to cruising. Constant-power acceleration is more realistic, but to present the results with normalized quantities here we assume constant acceleration.

We will compare the air drag and tire rolling energies in two extreme scenarios. The first extreme scenario (standing start) is to accelerate as slowly as possible so that all of the distance (t_{total}) is covered at constant acceleration. The second extreme scenario (flying start) is to accelerate so quickly that practically all of the distance is covered at constant speed.

When we do the math we find that the energy ratio between these extreme scenarios (energy at constant acceleration to energy at constant speed) is 2:1 for air drag energy and 4:3 for tire rolling energy (Equations 12 and 17). The second extreme scenario (flying start, constant speed) uses the least amount of energy that is possible to cover a given distance (d_{total}) at a given time (t_{total}). This scenario can be thought of as infinite acceleration over an infinitesimally short distance at the start (acceleration distance = 0). In the first extreme scenario, constant acceleration is maintained over the entire distance (acceleration distance = 1). We can simulate intermediate scenarios as the normalized acceleration distance is varied from 0 (flying start) to 1 (standing start), and show that the fastest acceleration requires the least energy. For air drag energy, this result is plotted in Figure 3.

In Figures 1, 2 and 3, all quantities are normalized with respect to reference quantities as follows: the reference speed is the maximum speed with constant acceleration over the entire distance ($d_a = d_{total}$). In this condition the average speed is half the maximum speed. Drag force, drag power and drag energy referring to this condition are all normalized as 1.0 in all graphs (Figures 1, 2 and 3). For example, a value of

0.7 on the horizontal axis indicates 70% of the total distance; and for any quantity (speed, power, force, energy), a value 0.2 on the vertical axis indicates 20% of the maximum value of that quantity in the constant acceleration regime.

In Figure 2, the accelerating distance is arbitrarily set to 0.2 as an example, and the acceleration is chosen such that the average speed at the end of the total distance becomes 0.5 (to match its counterpart in Figure 1). It can be seen that in this case the total air drag energy is only 64% of its counterpart in Figure 1. In other words, by accelerating quickly over 20% of the distance and cruising over the rest of it (instead of accelerating slowly over all of it), the same distance is covered at the same time with 36% less air drag energy. Tire rolling energy (not shown) is also reduced.

With even higher acceleration rates (i.e., shorter acceleration distances), even more energy can be saved for the same average speed. This relationship is shown in Figure 3. Mathematically, the energy saving would reach 50% when the acceleration distance approaches its theoretical limit of 0. Energy savings accomplished by high power are offset, at least in part, by extra heat that dissipates from the battery, motor and controller at high power. The energy we seek to minimize includes this heat. These losses are analyzed under the “heat” heading of this article.

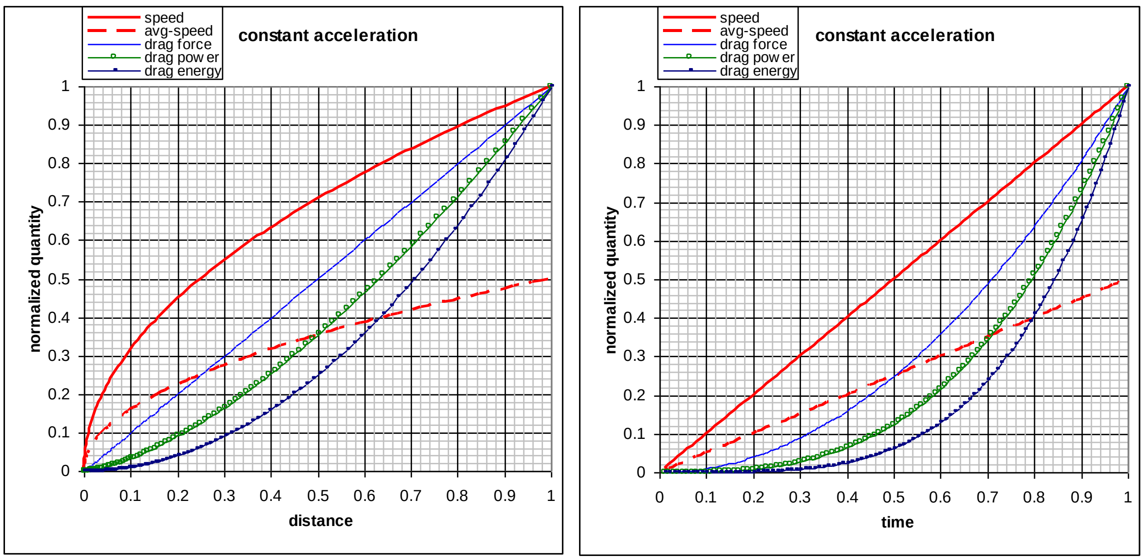

Figure 1 –constant acceleration (reference condition); accelerating distance = 1.0. Maximum speed = 1.0. Average speed = 0.5. Total drag energy = 1.0. Note that drag force is proportional to distance, and speed is proportional to time.

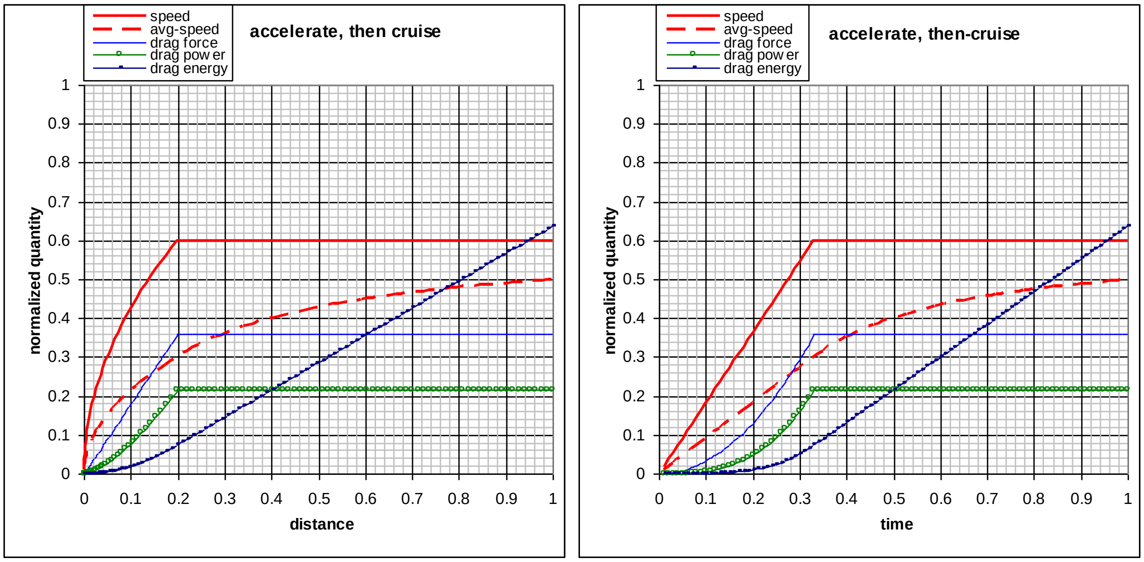

Figure 2 –constant acceleration followed by constant speed; accelerating distance = 0.2. Maximum speed = 0.6. Average speed = 0.5. Total drag energy = 0.64

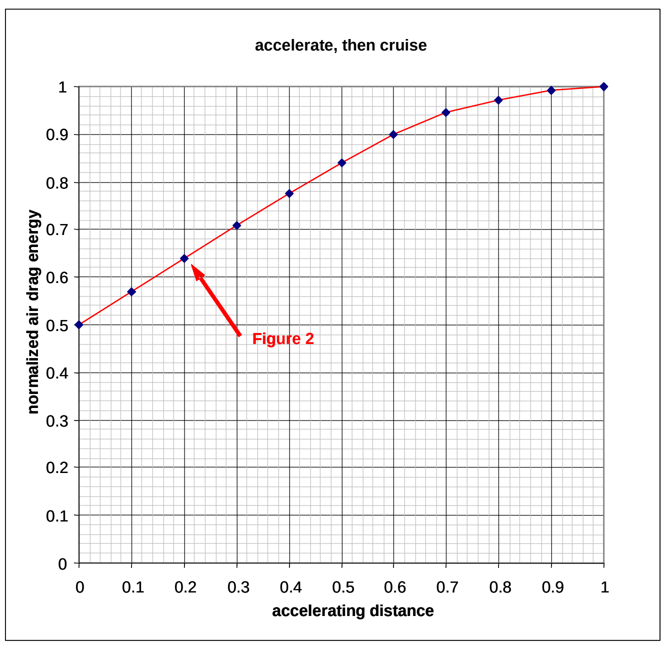

Figure 3 –normalized air drag energy as a function of accelerating distance d_a. The point representing Figure 2 (0.2, 0.64) is indicated with the arrow.

We have seen that the energy-wasting properties of hills and accelerations are rooted in the resistive natures of air drag and tire rolling resistance. Derivations of the equations for these loads are given at the end of the article. Equations 1 and 13 are used for calculating the resistive loads. The relative energies that these loads dissipate at constant acceleration and constant speed are given by Equations 12 and 17. Other equations are derived to calculate the effect of winds and Inertia.

HEAT

Heat is a form of energy; and when minimizing the total energy, heat losses must also be accounted for.

Heat losses at the load side of the energy equation

When mechanical power is transferred from the source to the load, some mechanical energy is dissipated as heat, for example by friction in the chain or gears, and by hysteresis losses in muscles due to frame deformations. In our simulation model, we account for these losses by specifying drivetrain efficiency. Drivetrain loss is calculated by dividing the output power by the drivetrain efficiency. In the present version of the simulation model, drivetrain efficiency is modeled as a constant. However, if engineering data is available, its value can be specified as a function of power. For most vehicle designs, drivetrain efficiency generally increases with power, up to a point.

Heat losses at the source side of the energy equation

So far, all of our analysis has been focused on minimizing the mechanical energy at the load side of the energy equation. On the source side, energy is dissipated in the form of heat from the battery, wiring, controller and motor. When calculating the electric power drawn from the battery, these heat-dissipation rates must be added to the output power of the electric motor. Heat is also dissipated from the body of the rider, but since our goal is to minimize *electric* energy, our equations include only the mechanical *output* power of the rider, and we are not concerned with the efficiency of the conversion between food energy and mechanical energy.

Heat losses in the motor:

In the simulation model the motor's heat dissipation rate is calculated by dividing the motor's output power by its efficiency. In the design of the vehicle, engineering accommodations should be made such that the motor always operates near its peak efficiency for the particular power it is producing. In that case, motor efficiency can be expressed directly as a function of motor power; and the present version of the simulation model allows only for a fixed relationship between motor efficiency and motor power. But when engineering parameters, such as voltages, currents and motor speeds are known, motor efficiency for each condition can be calculated from these parameters.

Heat losses in the battery:

As electrical current flows from the battery to the motor, some of the energy is released as heat. The time rate at which heat is dissipated is the heat power. The nature of batteries is such that as electric power is increased, the ratio of heat power to electric power increases. To calculate heat power it is not necessary to know the battery's temperature. The simulation model calculates the battery's heat power by dividing its electric output power by its power efficiency. A given battery's power efficiency under different discharge rates can be extrapolated from its specifications in the form of Amp-hours versus C-rate. This heat power is added to the total resistive power. Thus, during climbs and accelerations, while increasing electric power saves energy (over the entire course) at the load side of the energy equation by decreasing air turbulence, it wastes energy at the source side by increasing the heat dissipation from the battery. Instantaneous electric power must be regulated to minimize the net energy including both sides of the energy equation over the entire course.

Heat losses in the controller and wiring are similarly calculated by dividing output power by efficiency. The efficiency of the battery and controller (including the wiring) may be specified as a unit.

DESIGN RECOMMENDATIONS

The need to achieve high rates of acceleration and elevation gain calls for a high power-to-weight ratio, which can be achieved with lightweight frames and components. The need for high efficiency motors at high power combined with low weight suggests the use of high voltage, low current, high rpm motors with reduction gears, but other suitable engineering solutions may also be found. Lightweight electric motors developed for RC-racing (radio-controlled model vehicle racing) may be adapted to electric bicycles. The mechanical design of the vehicle should be considered together with the electrical system. For example, motors that drive the crankset can be more efficient than hub motors by virtue of operating over a narrow range of speeds. On the other hand, AC induction motors with advanced controls can maintain near-maximum efficiency over a wider range of speed and torque conditions than DC brushless motors controlled by pulse width modulation (PWM). Alternately, multiple gears or continuously variable transmissions (CVT) can be used to keep the motor operating more nearly at peak efficiency under variable power demands.

DERIVATION OF EQUATIONS

Power = Force * speed
Energy = Force * distance

Air drag

In the absence of wind, air drag is proportional to the square of the bike speed, and is given by the following equation:

$$F_{air} = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \cdot u^2 \qquad (1) \quad \text{air drag force}$$

where

F_{air} = drag force
ρ = density of air
C_d = aerodynamic drag coefficient
A = frontal area
u = bike speed

In Equation (1), u is the instantaneous speed of the bicycle, which, in general, can be varied according to a schedule. The energy (product of force and distance) to cover a given distance in a given time depends on the schedule. To demonstrate this, we derive equations for air drag energies with two different schedules that cover the same distance (D) at the same time (T). The first schedule is constant acceleration from a standstill; the second schedule is constant speed.

AIR DRAG ENERGY with CONSTANT ACCELARATION (E_{air_ca})

t = time
x = distance
u = speed (variable)
a = acceleration (constant).
when t=0, x=0 and u=0 (initial conditions when acceleration begins)
when t=T, x=D and u=U (final conditions when acceleration ends)

$$u = a \cdot t \qquad (2) \quad \text{speed}$$

$$x = \frac{1}{2} \cdot a \cdot t^2 \qquad (3) \quad \text{distance}$$

$$x = \frac{u^2}{2 \cdot a} \qquad (4) \quad \text{from 3 and 2; eliminating t}$$

$$u_{avg} = \frac{x}{t} = \frac{\left(\frac{u^2}{2 \cdot a}\right)}{\left(\frac{u}{a}\right)} = \frac{u}{2} \qquad (5) \quad \text{average speed; from 4 and 2}$$

Note that the average speed is determined from instantaneous speed regardless of the rate of acceleration, as long as it is constant.

Average speed from start (x=0) to end (x=D)

$u_{avg} = U / 2$ (5)

Air drag energy

$$dE = F(x) \cdot dx$$ (6) energy required to move distance dx

$$E = \int \left[\left(\frac{1}{2} \cdot \rho \cdot C_d \cdot A \right) \cdot (2 \cdot a \cdot x) \right] \cdot dx$$ (7); substituting 1 and 4 into 6

$$E_{air\_ca} = \frac{1}{2} \cdot a \cdot \rho \cdot C_d \cdot A \cdot D^2$$ (8); integrating 7 from x=0 to x=D.

AIR DRAG ENERGY with CONSTANT SPEED (E_{air_cs})

t = time
x = distance
u = speed = (x / t) = constant.

when t=0, x=0 and u = U_{avg} initial condition

when t=T, x=D and u = U_{avg} condition for matching the distance of constant acceleration

$$u = U_{avg} = \frac{U}{2}$$ condition for matching the average speed of constant acceleration (9)

$$E = F_{air} \cdot D$$ air drag (force) is constant over distance D

$$E_{air\_cs} = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \cdot \left(\frac{U^2}{4} \right) \cdot D$$ from 1 and 9

$$E_{air\_cs} = \frac{1}{2} \cdot \rho \cdot C_d \cdot A \cdot \left(\frac{2 \cdot a \cdot D}{4} \right) \cdot D$$ (10) from 9 and 4

$$E_{air\_cs} = \frac{1}{4} \cdot a \cdot \rho \cdot C_d \cdot A \cdot D^2$$ (11) Air drag energy from start to end

$$E_{air\_ca} = 2 \cdot E_{air\_cs}$$ (12) from 11 and 8

Equations (5) and (12) show that when an object at rest moves to a given distance at constant acceleration, it reaches twice the speed and dissipates twice the air-drag energy as when covering the same distance at the same time at constant speed (Figure 1). Therefore, if the only movement choices were constant acceleration and constant speed, accelerating over the smallest portion of a distance and consequently covering the longest portion at constant speed would result in dissipating the smallest amount of air-drag energy. This means that in order to reach a given destination at a given time with minimum air-drag energy, accelerations should be avoided (i.e., maintain a constant speed in spite of changing load conditions); and if that is not possible, accelerations should be done at the highest rates (i.e., over shortest distances) possible.

Tire rolling resistance

Tire rolling resistance is proportional to the product of the total weight W and the bike speed u, and its value is given by Equation 13.

$$F_{tire} = W \cdot \sqrt{(1 - incline^2)} \cdot C_{rr} \cdot u \qquad \text{(13) tire rolling resistance}$$

C_{rr} is called the coefficient of tire rolling resistance. If the front and rear tires are known to have different coefficients, then C_{rr} is calculated from those values and the weight distribution.

For a given average speed over a given distance, we'll calculate tire rolling energies dissipated in constant-acceleration and constant-speed scenarios, just as we did for air-drag energies with Equations 8 and 11. For moderate hills, the effect of the incline on C_{rr} is negligibly small; hence, in Equation 13 we assume that incline = 0. Again we find that, like air drag energy, tire rolling energy is also greater with constant acceleration than with constant speed, and the ratio between them is also independent of the rate of acceleration (Equation 17).

TIRE ROLLING ENERGY with CONSTANT ACCELARATION (at x=0, u=0; at x=D, u=U = 2*U_{avg})

$$dE = F(x) \cdot dx = W \cdot C_{rr} \cdot u \cdot dx \qquad \text{(14) tire rolling energy required to move distance dx}$$

By integrating Equation 14 from x=0 to x=D and substituting 2*a*x for u^2 we obtain

$$E_{tire\_ca} = \frac{2}{3} \cdot W \cdot C_{rr} \cdot U \cdot D \qquad \text{(15) tire rolling energy (constant acceleration)}$$

TIRE ROLLING ENERGY with CONSTANT SPEED (u = constant = U/2)

$$dE = F \cdot dx = W \cdot C_{rr} \cdot \frac{U}{2} \cdot dx$$

$$E_{tire\_cs} = \frac{1}{2} \cdot W \cdot C_{rr} \cdot U \cdot D \qquad \text{(16) tire rolling energy (constant speed)}$$

$$E_{tire\_ca} = \frac{4}{3} \cdot E_{tire\_cs} \qquad \text{(17) from 15 and 16}$$

When we write the expressions for *total* dissipated energies (sum of tire-rolling and air-drag energies) for constant acceleration and constant speed scenarios of equal average speed, we find that the ratio between the two energies cannot be made independent of physical constants such as weight, frontal area, tire coefficients, etc. Nevertheless, from Equations 18 and 19 we can conclude that constant speed will always dissipate less energy than constant acceleration, because with any set of positive constants both terms of equation 19 will each have smaller values than their counterparts in equation 18.

$$E_{dissipated\_ca} = (E_{air\_ca} + E_{tire\_ca}) =$$
$$= \left(\frac{1}{2} \cdot a \cdot \rho \cdot C_d \cdot A \cdot D^2 \right) + \left(\frac{2}{3} \cdot W \cdot C_{rr} \cdot \sqrt{2a} \cdot \sqrt{D^3} \right) \qquad \text{(18) from 8 and 15}$$

and,

$$E_{dissipated\_cs} = (E_{air\_cs} + E_{tire\_cs}) =$$
$$= \left(\frac{1}{4} \cdot a \cdot \rho \cdot C_d \cdot A \cdot D^2 \right) + \left(\frac{1}{2} \cdot W \cdot C_{rr} \cdot \sqrt{2a} \cdot \sqrt{D^3} \right)$$ (19) from 11 and 16

CONSTANT POWER ACCELERATION

In the above calculations, the rate of acceleration is treated as a constant. Because resistive loads (air drag and tire rolling resistance) increase with speed, constant acceleration dictates that during the acceleration, power must increase with time and distance. While this assumption makes calculations easier, in practice, especially for a bicycle, constant acceleration is not realistic. Constant power acceleration is a more reasonable assumption. With constant power acceleration, the speed profile (constant-power equivalent of Figure 1) would depend on the value of moving mass relative to frontal area and the coefficients of air drag and tire rolling resistance. With the help of the simulation model, we can calculate the energy with constant-power acceleration for a given set of these constants. When we do such calculations we see that, accelerating with the highest available power over the shortest distance possible (and consequently covering the longest distance at constant speed), will minimize the energy dissipated by air drag and tire rolling resistance.

Inertia force

The force required to overcome inertia is calculated from the formula known as Newton's Second Law

F = m * a   where m is the mass being accelerated and a is the rate of acceleration.

Compared to motor vehicles, a person riding a bicycle has very small mass, very little power, a relatively large frontal area and a large drag coefficient. As a consequence, when a bicycle accelerates from a standstill to a given speed, the distance is relatively long, and a relatively large portion of the energy is spent against air drag. Equations 1 through 12 are for calculating that portion of the energy. We also want to calculate the inertia portion. Because the acceleration starts from rest, this energy is equal to the kinetic energy at the end of acceleration.

$$E = \frac{1}{2} \cdot m \cdot U^2$$ kinetic energy at the end of acceleration

$$U^2 = 2 \cdot a \cdot D$$ from Equation (4)

$$E_{inertia\_ca} = m \cdot a \cdot D$$ (20) energy spent against inertia at constant acceleration

Note that with constant acceleration, the inertia energy is proportional to distance (Equation 20); but air-drag energy is proportional to distance-squared (Equation 8).

Winds

In Equation (1) is for an object moving in still air. But air drag is also a function of the wind speed and wind direction. If a cyclist is moving east, and the wind is from the north, then the wind velocity, as perceived by the moving cyclist, will be coming from somewhere in the northeast quadrant. Air drag must be computed from this relative wind, called the apparent wind, rather than from the bike speed on the ground. Furthermore, for bicycles and other low-power vehicles, ground speed itself depends on the air drag; so, the apparent wind speed, apparent wind direction, and bike speed, must be calculated together from equations containing absolute wind speed and direction, available power, and physical constants.

Figure 4 is a birds-eye view of a cyclist and a vector diagram to show the bike and wind velocities.

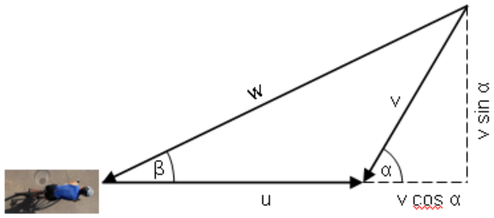

Figure 4 –Vector diagram, bird's-eye view of cyclist riding in a partial cross wind.

u = bike speed
v = wind speed
α = wind angle (0° = headwind)
w = apparent wind speed (wind relative to the moving cyclist)
β = apparent wind angle

Apparent wind conditions w and β are related to absolute wind conditions v, α and u, by Equations 21 and 22 (refer to Figure 6).

$$w^2 = u^2 + v^2 + 2 \cdot u \cdot v \cdot \cos\alpha \qquad (21)$$

$$\cos\beta = \frac{u + v \cdot \cos\alpha}{w}$$

$$\cos\beta = \frac{u + v \cdot \cos\alpha}{\sqrt{u^2 + v^2 + 2 \cdot u \cdot v \cdot \cos\alpha}} \qquad (22)$$

When there is no wind (v=0; α = β =0; w = u), the apparent wind speed equals the ground speed and thus air drag can be calculated from Equation (1). We would like to calculate the air drag in the more general case when air velocity and ground velocity have different magnitudes and directions (Equation 25).

From Figure 6 and Equations 21 and 22 it is evident that when v is not zero, u and w are different and β is also not zero. But in addition, the effective area presented to the apparent wind is generally different from the 'frontal' area A. The effective air drag coefficient is also generally different from the frontal air drag coefficient C_d. To account for these effects, the drag area is multiplied by a scalar λ. Thus, the aerodynamic force becomes

$$F_{air} = \frac{1}{2} \cdot \rho \cdot C_d \cdot (\lambda \cdot A) \cdot w^2 \qquad (23)$$

This force is aimed at angle β to the axis of movement on the ground. To quantify λ, another coefficient μ, called the aerodynamic aspect ratio, is defined. This coefficient is the 'crosswind to headwind drag force ratio'. This ratio doesn't depend on the speed or direction of the object moving in air, but is a characteristic of the *shape* of the object. It could be determined experimentally by placing the object (or a scale model of it), perpendicular and parallel, respectively, to the airflow (for example in a wind tunnel), and measuring the ratio between the two drag forces. In the simulation model, μ is a user-defined constant. In the examples presented here, the default value of μ = 1.2 is used, which is estimated to be appropriate for the shape of a person riding a bicycle. In Equation (23), λ is calculated from μ and β as follows:

$$\lambda = \cos^2\beta + \mu \cdot \left(\sin^2\beta\right)$$

This function is chosen simply because it satisfies the headwind and crosswind conditions (λ=1 when β=0°, and λ=μ when β=90°, respectively), and has a smooth transition between them. Other functions or lookup tables may be used if aerodynamic data are available for the shape of the vehicle.

In terms of independent variables u, v and α, the expression for λ becomes

$$\lambda = \frac{(u + v \cdot \cos\alpha)^2}{u^2 + v^2 + 2 \cdot u \cdot v \cdot \cos\alpha} + \mu \cdot \left(1 - \frac{(u + v \cdot \cos\alpha)^2}{u^2 + v^2 + 2 \cdot u \cdot v \cdot \cos\alpha}\right) \qquad (24)$$

While the aerodynamic force on the moving object is F_{air} (Equation 23), the actual drag force (the component of F_{air} projected against the direction of travel) is

$$F_{drag} = F_{air} \cdot \cos\beta = \frac{1}{2} \cdot \rho \cdot C_d \cdot \lambda \cdot A \cdot w^2 \cdot \cos\beta \qquad (25)$$

The energy required to drag this force over a distance x is

$$E_{drag} = F_{drag} \cdot x = \frac{1}{2} \cdot \rho \cdot C_d \cdot \lambda \cdot A \cdot w^2 \cdot \cos\beta \cdot x \qquad (26)$$

And the air drag power is

$$P_{drag} = F_{drag} \cdot u = \frac{1}{2} \cdot \rho \cdot C_d \cdot \lambda \cdot A \cdot w^2 \cdot \cos\beta \cdot u \qquad (27)$$

Equation 27 is written in terms of apparent wind speed w and apparent wind angle β. By substituting for w^2 and cos β, their expressions from Equations 21 and 22, we obtain air drag power in terms of the independent variables bike speed u, wind speed v and wind angle α:

$$P_{drag} = \frac{1}{2} \cdot \rho \cdot C_d \cdot \lambda \cdot A \cdot \sqrt{u^2 + v^2 + 2 \cdot u \cdot v \cdot \cos\alpha} \cdot (u + v \cdot \cos\alpha) \cdot u \qquad (28)$$

To calculate how much power to apply to minimize the total energy for a given average speed in arbitrary wind conditions we can use the simulation model that includes this equation. The value of λ is given in Equation 24.

<u>Hill force</u>

The expression for the hill force is [F_{hill} = weight * incline], where incline is defined as the elevation gain per distance traveled on the inclined road surface.

Special note: this definition of incline, which is the sine of the angle that the road makes with horizontal, is different from the standard definition of grade (rise over run -the tangent of the same angle), which is used in railroad and highway engineering. For grades less than 12%, this difference is not significant.

TOTAL POWER

If the bike speed, wind speed and wind angle were known, the air-drag force can be calculated from Equation (28). But in practice, bike speed cannot be an independent variable. For example, bike speed is naturally higher going downhill than uphill. It is also higher with tailwinds than with headwinds. Due to the limitation of available power, bike speed is determined primarily by incline and air drag. In turn, a change in bike speed results in a change in the magnitude and angle of the apparent wind, which affects air drag. And air drag determines bike speed. Therefore, instead of assuming a bike speed, we must *calculate* it from independent variables; namely, available power, wind speed, and wind angle.

Equation (28) gives only the air drag power. Bike speed must be calculated from the total power. Total power includes the powers consumed by other loads, namely, climbing force, tire rolling resistance and inertia. The equations for these powers are given below.

Climbing power

$$P_{climb} = (W \cdot incline) \cdot u \quad (30)$$

Where W is weight and *incline* = elevation gain per distance on the road

Tire rolling power

$$P_{tire} = W \cdot (1 - incline^2) \cdot C_{rr} \cdot u \quad (31)$$

where Crr is the coefficient of tire rolling resistance

Inertia power

$$P_{inertia} = m \cdot a \cdot u \quad (32)$$

where m is mass, a is acceleration and u is bike speed.

Power is also dissipated at the source, as heat, for example in the coil windings of an electric motor, or in a battery. Thus, the total power, including the load and the source is

$$P_{total} = P_{climb} + P_{inertia} + P_{drag} + P_{tire} + P_{heat} \quad (33)$$

P_{total} is the instantaneous power that changes along the course. In “constant power” mode of analysis, P_{total} is assigned a given value and bike speed u is found as the smallest real positive root of Equation 33.